

Another View on the Hölder Inequality

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Abstract

Every diagonal matrix \mathbf{D} yields an endomorphism on the n -dimensional complex vector space. If one provides the \mathbb{C}^n with Hölder norms, we can compute the operator norm of \mathbf{D} . We define homogeneous weighted spaces as a generalization of normed spaces. We generalize the Hölder norms for negative values, this leads to a proof of an extended version of the Hölder inequality. Finally, we formulate this version also for measurable functions.

1 Introduction

In this paper we generalize the well-known Hölder inequality (see, for instance, [1] or [2], or other books on functional analysis). So far nobody discussed the case of negative exponents in all details (for some discussions see e.g. [3], p.51). The main reason for this might be the fact that for $p < 0$ the map $(x_1, x_2, \dots, x_n) \mapsto \sqrt[p]{|x_1|^p + |x_2|^p + \dots + |x_n|^p}$ does not yield a norm for \mathbb{C}^n , because it is neither positive definit, nor the triangle inequality holds. Although it is worth to consider this map, since this leads to a natural extension of the often used Hölder inequality. To get this result, we first introduce *homogeneous weighted spaces* generalizing normed spaces. Then we define *Hölder weights* as a generalization of the Hölder norms, and the *operator weight* as a generalization of the operator norm. In our first rather inconvenient theorem we compute the operator weight of a diagonal matrix. The main result of this paper is then an extension of the Hölder inequality. Finally, we prove an analogic result for measurable functions. But here the proofs rely on the standard Hölder inequality.

Let X be a complex vector space. Let $\|\cdot\|$ denote a positive functional on X , that means: $\|\cdot\|: X \longrightarrow \mathbb{R}^+ \cup \{0, \infty\}$. We consider three conditions,

- (1) $\|\vec{0}\| = 0$ and for all $z \in \mathbb{C}$ and all $\vec{x} \in X$ we have: $\|z \cdot \vec{x}\| = |z| \cdot \|\vec{x}\|$ ("homogeneity"),
- (2) $\infty \notin \text{image}(\|\cdot\|)$ and $\|\vec{x}\| = 0$ if and only if $\vec{x} = \vec{0}$ ("positive definiteness"),
- (3) For all $\vec{x}, \vec{y} \in X$ one has $\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$ ("triangle inequality").

Definition 1.

- If $\|\cdot\|$ fullfils (1) then we call $\|\cdot\|$ a *homogeneous weight* on X ,
- if $\|\cdot\|$ fullfils (1), (2) then we call $\|\cdot\|$ a *pseudonorm* on X , and
- if $\|\cdot\|$ fullfils (1), (2) and (3) then $\|\cdot\|$ is called a *norm* on X .

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According to these three cases we call the pair $(X, \|\cdot\|)$ a homogeneously weighted vector space (or **hw space**), a pseudonormed vector space, or a normed vector space, respectively.

Definition 2. For a linear map $F : (X, \|\cdot\|_X) \rightarrow (Y, \|\cdot\|_Y)$ between complex homogeneously weighted vector spaces we denote by $\|F\| := \inf \{C > 0 \mid \forall \vec{x} \in X : \|F(\vec{x})\|_Y \leq C \cdot \|\vec{x}\|_X\}$ the *operator weight* of F with respect to $\|\cdot\|_X, \|\cdot\|_Y$.

Let \mathbf{A} be a complex valued $m \times n$ matrix, $m, n \in \mathbb{N}$. Then \mathbf{A} defines a linear map, $\mathbf{A} : \mathbb{C}^n \rightarrow \mathbb{C}^m$. Let $\|\cdot\|_X, \|\cdot\|_Y$ be homogeneous weights on $X := \mathbb{C}^n$ and $Y := \mathbb{C}^m$, respectively. Then the operator weight is $\|\mathbf{A}\| = \inf \{C > 0 \mid \forall \vec{x} \in \mathbb{C}^n : \|\mathbf{A}\vec{x}\|_Y \leq C \cdot \|\vec{x}\|_X\}$.

This definition turns $\{\mathbf{A} \mid \mathbf{A} : (\mathbb{C}^n, \|\cdot\|_X) \rightarrow (\mathbb{C}^m, \|\cdot\|_Y) \text{ and } \mathbf{A} \text{ is linear}\}$ into a **hw space**, which is a pseudonormed space, or a normed space, respectively, depending on the properties of the homogeneous weights $\|\cdot\|_X$ and $\|\cdot\|_Y$.

Now for every $n \in \mathbb{N}$ and for every $p \in \{\infty, -\infty\} \cup \mathbb{R} \setminus \{0\}$ we construct a homogeneous weight on \mathbb{C}^n .

Definition 3.

For $\vec{x} = (x_1, \dots, x_n) \in \mathbb{C}^n$ and for $p \in (0, \infty)$ set $\|\vec{x}\|_p := \sqrt[p]{|x_1|^p + |x_2|^p + \dots + |x_n|^p}$, and for $p \in (-\infty, 0)$ we set

$$\|\vec{x}\|_p := \begin{cases} \sqrt[p]{|x_1|^p + |x_2|^p + \dots + |x_n|^p} & \text{for } \prod_{i=1}^n x_i \neq 0 \\ 0 & \text{for } \prod_{i=1}^n x_i = 0 \end{cases}$$

and $\|\vec{x}\|_\infty := \max\{|x_i| \mid i \in \{1, 2, \dots, n\}\}$, and $\|\vec{x}\|_{-\infty} := \min\{|x_i| \mid i \in \{1, 2, \dots, n\}\}$. These homogeneous weights will be called the *Hölder weights* on \mathbb{C}^n .

Remark 1. Note that for $p < 0$ we have $\|\vec{x}\|_p = 0 \iff \exists j \in \{1, \dots, n\} \text{ and } x_j = 0$. Furthermore, for all $n > 1$, these Hölder weights are pseudonorms if and only if $p > 0$, and they are norms if and only if $p \geq 1$.

In the case of a diagonal matrix \mathbf{D} , $\mathbf{D} : (\mathbb{C}^n, \|\cdot\|_s) \rightarrow (\mathbb{C}^n, \|\cdot\|_t)$ and $\|\cdot\|_s, \|\cdot\|_t$ are Hölder weights, one easily verifies that

$$\|\mathbf{D}\| = \sup \{\|\mathbf{D}\vec{x}\|_t \mid \vec{x} \in \mathbb{C}^n \text{ and } \|\vec{x}\|_s = 1\}.$$

This equality does not hold in general for arbitrary linear maps $F : (X, \|\cdot\|_X) \rightarrow (Y, \|\cdot\|_Y)$ due to the fact that there need not to exist an \vec{x} with $\|\vec{x}\|_X = 1$.

Let us now restrict our attention to diagonal matrices to state our first theorem.

Theorem 1. For $n \geq 2$ and $\vec{v} := (v_1, \dots, v_n) \in \mathbb{C}^n$ let $\mathbf{D} := \begin{pmatrix} v_1 & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & v_n \end{pmatrix}$

be the associated n -dimensional diagonal matrix, and let $s, t \in \mathbb{R} \setminus \{0\} \cup \{+\infty, -\infty\}$. Thus \mathbf{D} is a linear endomorphism on \mathbb{C}^n . Then we have for the operator weight $\|\mathbf{D}\|$ with respect to $\|\cdot\|_s$ and $\|\cdot\|_t$

$$\|\mathbf{D}\|_{s,t} := \|\mathbf{D}\| = \begin{cases} \infty & \text{if } (s < 0 < t \wedge \vec{v} \neq \vec{0}) & \text{(A),} \\ \|\vec{v}\|_t & \text{if } (\vec{v} = \vec{0}) \vee (t < 0 \wedge \prod_{i=1}^n v_i = 0) \vee (s = \infty) & \text{(B),} \\ \|\vec{v}\|_{\frac{s \cdot t}{s-t}} & \text{if } (-\infty < t < s < 0 \wedge \prod_{i=1}^n v_i \neq 0) \vee \\ & (0 < t < s < \infty) \vee (-\infty < t < 0 < s < \infty) & \text{(C),} \\ \|\vec{v}\|_\infty & \text{if } (s \leq t < 0 \wedge \prod_{i=1}^n v_i \neq 0) \vee (0 < s \leq t) & \text{(D),} \\ \|\vec{v}\|_{-s} & \text{if } (t = -\infty \wedge \prod_{i=1}^n v_i \neq 0) & \text{(E).} \end{cases}$$

Note that all possible cases are covered by (A) – (E). The above theorem allows us to deduce a theorem and two corollaries.

Corollary 1. *Let $s, t \in \mathbb{R}$ such that $0 \neq s \cdot t$, and for $\mathbf{D} := \text{diag}(v_1, \dots, v_n)$ with $\prod_{i=1}^n v_i \neq 0$ we have*

$$\|\mathbf{D}\|_{s,t} = \|\mathbf{D}\|_{-t,-s}.$$

Theorem 2. [Generalized Hölder Inequality]

Let $r, s, t \in \mathbb{R}$ and $0 \neq r \cdot s \cdot t$ and $\frac{1}{t} = \frac{1}{r} + \frac{1}{s}$. Then we have for every $n \in \mathbb{N}$ for all vectors $\vec{v} := (v_1, \dots, v_n)$ and $\vec{x} := (x_1, \dots, x_n) \in \mathbb{C}^n$ (with $\vec{v} \cdot \vec{x}$ denotes multiplication by components)

$$t < r, s \implies \|\vec{v} \cdot \vec{x}\|_t \leq \|\vec{v}\|_r \cdot \|\vec{x}\|_s,$$

$$t > r, s \implies \|\vec{v} \cdot \vec{x}\|_t \geq \|\vec{v}\|_r \cdot \|\vec{x}\|_s.$$

More explicitly we have the following corollary.

Corollary 2. [Generalized Hölder Inequality]

Let $r, s, t \in \mathbb{R}$ such that $0 \neq r \cdot s \cdot t$ and $\frac{1}{t} = \frac{1}{r} + \frac{1}{s}$. Then for every $n \in \mathbb{N}$ and for all numbers $v_1, \dots, v_n, x_1, \dots, x_n \in \mathbb{C}$ with $\prod_{i=1}^n v_i \cdot x_i \neq 0$ we have

$$t < r, s \implies \sqrt[t]{\sum_{i=1}^n |v_i \cdot x_i|^t} \leq \sqrt[r]{\sum_{i=1}^n |v_i|^r} \cdot \sqrt[s]{\sum_{i=1}^n |x_i|^s},$$

$$t > r, s \implies \sqrt[t]{\sum_{i=1}^n |v_i \cdot x_i|^t} \geq \sqrt[r]{\sum_{i=1}^n |v_i|^r} \cdot \sqrt[s]{\sum_{i=1}^n |x_i|^s}.$$

Remark 2. If $\prod_{i=1}^n v_i \cdot x_i = 0$ the inequality remains true provided the roots for negative exponents are defined.

2 Proof of Theorem 1

First we handle the two easy cases.

CASE (A). Let $s < 0 < t$ and $\vec{v} \neq \vec{0}$.

Because \mathbf{D} is not the 0-matrix, there is a $j \in \{1, \dots, n\}$ with $v_j \neq 0$. Take for every $k \in \mathbb{N} \setminus \{1\}$ the vector $\vec{a}_k := (a_{k,1}, \dots, a_{k,n})$ with $a_{k,j} := k$ and for all $i \in \{1, 2, \dots, n\} \setminus \{j\}$ let $a_{k,i} := \sqrt[s]{\frac{1-k^s}{n-1}}$. We have for every $k \in \mathbb{N} \setminus \{1\}$: $\|\vec{a}_k\|_s = 1$ and $\|\mathbf{D}(\vec{a}_k)\|_t \geq |k \cdot v_j|$, and because of $k \rightarrow \infty$ the right hand side goes to infinity, hence $\|\mathbf{D}\|_{s,t} = \infty$.

CASE (B). Let $\vec{v} = \vec{0}$, or $t < 0$ and $\prod_{i=1}^n v_i = 0$, or $s = \infty$.

If \mathbf{D} is the 0-matrix we have for all s, t : $\|\mathbf{D}\|_{s,t} = 0$. If $(t < 0 \wedge \prod_{i=1}^n v_i = 0)$ one has at least one $j \in \{1, \dots, n\}$ with $v_j = 0$. Then for $\vec{x} \in \mathbb{C}^n$ we have $v_j x_j = 0$, hence $\|\mathbf{D}(\vec{x})\|_t = 0$, hence $\|\mathbf{D}\|_{s,t} = 0 = \|\vec{v}\|_t$.

In the case of $s = \infty$ take $\vec{e} := (1, 1, \dots, 1)$, then we have $\|\vec{e}\|_\infty = 1$. If $t \in \mathbb{R}$ we get $\|\mathbf{D}\vec{e}\|_t = [\sum_{i=1}^n |v_i|^t]^{\frac{1}{t}}$. If $t = -\infty$ we get $\|\mathbf{D}\vec{e}\|_{-\infty} = \min\{|v_i| \mid i \in \{1, 2, \dots, n\}\}$. Hence in CASE (B) we always have $\|\mathbf{D}\|_{s,t} = \|\vec{v}\|_t$.

The following two cases are more complicated and they need more attention. They will be treated together, because the proofs are similar.

CASE (C) and CASE (D). Let either $(-\infty < t < s < 0 \wedge \prod_{i=1}^n v_i \neq 0)$, or $(0 < t < s < \infty)$, or $(-\infty < t < 0 < s < \infty)$, or $(s \leq t < 0 \wedge \prod_{i=1}^n v_i \neq 0)$, or $(0 < s \leq t)$.

The theorem is trivial if \mathbf{D} is the 0-matrix, because then it clearly follows that $0 = \|\vec{v}\|_{\frac{s \cdot t}{s-t}} = \|\vec{v}\|_\infty = \|\mathbf{D}\|$. Hence, we assume $\vec{v} \neq \vec{0}$. Let $\mathbf{M} \in \{1, 2, \dots, n\}$ such that $|v_{\mathbf{M}}| = \max\{|v_1|, \dots, |v_n|\}$, hence $|v_{\mathbf{M}}| > 0$. Now for the proof we will distinguish four different cases.

Case a) $0 < s, t < \infty$.

Case b) $-\infty < s, t < 0$ and $\prod_{i=1}^n v_i \neq 0$.

Case c) $-\infty < t < 0 < s < \infty$.

Case d) $(-\infty = s \leq t < 0 \text{ and } \prod_{i=1}^n v_i \neq 0) \text{ or } (0 < s \leq t = \infty)$.

We will prove the cases **a, b, c** for $n = 2$ and then inductively for all $n \in \mathbb{N} \setminus \{1\}$.

Case a) Let $0 < s, t < \infty$.

Let $n = 2$. We have the 2×2 matrix $\mathbf{D} := \text{diag}(v_1, v_2)$. Without loss of generality let $v_{\mathbf{M}} = v_2 (\neq 0)$. With $b := v_1/v_2$ we have $|b| \leq 1$, and $\mathbf{D} = v_2 \cdot \begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix} =: v_2 \cdot \tilde{\mathbf{D}}$.

We have $\|\mathbf{D}\|_{s,t} = |v_2| \cdot \|\tilde{\mathbf{D}}\|_{s,t} = |v_2| \cdot \sup \{ \|\tilde{\mathbf{D}}(\vec{x})\|_t \mid \vec{x} \in \mathbb{C}^2 \text{ and } \|\vec{x}\|_s = 1 \}$. With $\vec{x} := (x_1, x_2)$ we define a map $G : [0, 1] \rightarrow \mathbb{R}^+ \cup \{0\}$, but at first we will consider G^t because it is easier (G and G^t have extremums at the same values). Define

$G^t(y) := (\|\tilde{\mathbf{D}}(\vec{x})\|_t)^t = y^t \cdot |b|^t + [\sqrt[t]{1-y^s}]^t$ for $y := |x_1|$, $\|(x_1, x_2)\|_s = 1$, hence $y \in [0, 1]$. First assume that $s \neq t$. Elementary analysis shows that

$$(G^t)'(y_E) = 0 \Leftrightarrow y_E = \sqrt[s]{\frac{1}{1 + |b|^{\frac{s \cdot t}{t-s}}}}.$$

Instead of computing $(G^t)''(y_E)$ we check the boundaries of the domain of G , hence the maximum $M_{s,t} := \max \{ \|\tilde{\mathbf{D}}(\vec{x})\|_t \mid \vec{x} \in \mathbb{C}^2 \wedge \|\vec{x}\|_s = 1 \} = \max \{ G(y) \mid y \in [0, 1] \}$ is contained in the set $\{G(y_E), G(0), G(1)\} = \{ [1 + |b|^{\frac{s \cdot t}{s-t}}]^{\frac{s-t}{s \cdot t}}, 1, |b| \}$. To determine $M_{s,t}$ let us now consider the following three subcases.

Subcase 1: $s < t \Rightarrow M_{s < t} = 1$ and $\|\mathbf{D}\|_{s,t} = |v_2| \cdot M_{s < t} = |v_2|$.

Subcase 2: $s > t \Rightarrow M_{s > t} = [1 + |b|^{\frac{s \cdot t}{s-t}}]^{\frac{s-t}{s \cdot t}}$ and $\|\mathbf{D}\|_{s,t} = |v_2| \cdot M_{s > t} = [|v_2|^{\frac{s \cdot t}{s-t}} + |v_1|^{\frac{s \cdot t}{s-t}}]^{\frac{s-t}{s \cdot t}}$.

Subcase 3: $s = t \Rightarrow$ By doing similar calculations as just now (in the case $s \neq t$), we get $M_{s=t} = G(0) = 1$, hence $\|\mathbf{D}\|_{s,s} = |v_2|$, and the theorem has been proved for $n = 2$.

Remark 3. We have a continuous behaviour of $\|\mathbf{D}\|_{s,t}$ if $s = t$, that means

$$\lim_{\substack{t \nearrow s}} (\|\mathbf{D}\|_{s,t}) = \lim_{\substack{t \nearrow s}} ([|v_2|^{\frac{s \cdot t}{s-t}} + |v_1|^{\frac{s \cdot t}{s-t}}]^{\frac{s-t}{s \cdot t}}) = \|\vec{v}\|_\infty = |v_2| = \|\mathbf{D}\|_{s,s} = \lim_{\substack{s \searrow t}} (\|\mathbf{D}\|_{s,t}).$$

Proof for $n \geq 3$.

Assume that the theorem holds for $n-1$. Let $\mathbf{m} \in \{1, \dots, n-1\}$ with $|v_{\mathbf{m}}| := \max\{|v_1|, \dots, |v_{n-1}|\}$, let $\vec{x} := (x_1, \dots, x_{n-1}, x_n) \in \mathbb{C}^n$. We distinguish two subcases.

Subcase 1: $s < t$ or $s = t$.

We have just proved the theorem for $n = 2$, that means that for arbitrary $y_1, y_2, w_1, w_2 \in \mathbb{C}$ we have $\sqrt[t]{|w_1 y_1|^t + |w_2 y_2|^t} \leq \max\{|w_1|, |w_2|\} \cdot \sqrt[s]{|y_1|^s + |y_2|^s}$. By the assumption, we have $\sqrt[t]{\sum_{i=1}^{n-1} |v_i x_i|^t} \leq |v_{\mathbf{m}}| \cdot \sqrt[s]{\sum_{i=1}^{n-1} |x_i|^s}$.

By using the assumption and the theorem for $n = 2$, it follows that

$$\begin{aligned}
\|\mathbf{D}(\vec{x})\|_t &= \sqrt[t]{\sum_{i=1}^{n-1} |v_i x_i|^t + |v_n x_n|^t} \\
&\leq \sqrt[t]{|v_m|^t \cdot \left[\sqrt[s]{\sum_{i=1}^{n-1} |x_i|^s} \right]^t + |v_n x_n|^t} \\
&\leq \max\{|v_m|, |v_n|\} \cdot \sqrt[s]{\sum_{i=1}^{n-1} |x_i|^s + |x_n|^s} = |v_m| \cdot \|\vec{x}\|_s.
\end{aligned}$$

Hence $\|\mathbf{D}\|_{s,t} \leq |v_m|$.

The vector $\vec{e}_M := (0, \dots, 0, 1, 0, \dots, 0)$ shows that $\|\mathbf{D}(\vec{e}_M)\|_t = |v_m| \cdot 1$, hence $\|\mathbf{D}\|_{s,t} = |v_m|$.

Subcase 2: $s > t$.

Let $\tilde{w} := [\sum_{i=1}^{n-1} |v_i|^{\frac{s-t}{s-t}}]^{\frac{s-t}{s-t}}$. Because the theorem holds for $n = 2$, we have for arbitrary $y_1, y_2, w_1, w_2 \in \mathbb{C}$: $\sqrt[t]{|w_1 y_1|^t + |w_2 y_2|^t} \leq [|w_1|^{\frac{st}{s-t}} + |w_2|^{\frac{st}{s-t}}]^{\frac{s-t}{st}} \cdot \sqrt[s]{|y_1|^s + |y_2|^s}$.

Because we assume the theorem for $n - 1$, we have: $\sqrt[t]{\sum_{i=1}^{n-1} |v_i x_i|^t} \leq \tilde{w} \cdot \sqrt[s]{\sum_{i=1}^{n-1} |x_i|^s}$.

By using this and the theorem for $n = 2$, we have

$$\begin{aligned}
\|\mathbf{D}(\vec{x})\|_t &= \sqrt[t]{\sum_{i=1}^{n-1} |v_i x_i|^t + |v_n x_n|^t} \\
&\leq \sqrt[t]{\tilde{w}^t \cdot \left[\sqrt[s]{\sum_{i=1}^{n-1} |x_i|^s} \right]^t + |v_n x_n|^t} \\
&\leq [\tilde{w}^{\frac{st}{s-t}} + |v_n|^{\frac{st}{s-t}}]^{\frac{s-t}{st}} \cdot \|\vec{x}\|_s = \|\vec{v}\|_{\frac{s-t}{s-t}} \cdot \|\vec{x}\|_s.
\end{aligned}$$

Hence $\|\mathbf{D}\|_{s,t} \leq \|\vec{v}\|_{\frac{s-t}{s-t}}$.

Define for all $i = 1, 2, \dots, n$ $r_i := \sqrt[s-t]{|v_i|^t}$, and take the vector $\vec{z} := \frac{1}{\sqrt[s]{\sum_{i=1}^n |v_i|^{\frac{s-t}{s-t}}}} \cdot (r_1, \dots, r_n)$.

One has $\|\vec{z}\|_s = 1$ and $\|\mathbf{D}(\vec{z})\|_t = \|\vec{v}\|_{\frac{s-t}{s-t}}$, that means the theorem is satisfied both in subcase 1 and in subcase 2, and the proof is finished if $0 < s, t < \infty$.

Case b) Let $-\infty < s, t < 0$ and $\prod_{i=1}^n v_i \neq 0$.

Let $\mathbf{D} = \begin{pmatrix} v_1 & 0 \\ 0 & v_2 \end{pmatrix} = v_2 \cdot \begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix} =: v_2 \cdot \tilde{\mathbf{D}}$, with $b := v_1/v_2$, as above, and we have

$|v_2| \geq |v_1| > 0$ and $1 \geq |b| > 0$. One has $\|\mathbf{D}\|_{s,t} = |v_2| \cdot \sup\{\|\tilde{\mathbf{D}}(\vec{x})\|_t \mid \vec{x} \in \mathbb{C}^2 \wedge \|\vec{x}\|_s = 1\}$, as above.

But the domain of the map $G^t(y) := (\|\tilde{\mathbf{D}}(\vec{x})\|_t)^t = y^t \cdot |b|^t + [\sqrt[t]{1-y^s}]^t$ has changed.

With $\vec{x} = (x_1, x_2)$ and $\|\vec{x}\|_s = 1$, $y := |x_1|$, it has to be $y > 1$, (because s is negative).

As above, we have $(G^t)'(y_E) = 0 \Leftrightarrow y_E = \sqrt[s]{\frac{1}{1+|b|^{\frac{s-t}{t-s}}}} = [1 + |b|^{\frac{s-t}{t-s}}]^{-\frac{1}{s}}$,

and the maximum $M_{s,t} := \sup\{\|\tilde{\mathbf{D}}(\vec{x})\|_t \mid \vec{x} \in \mathbb{C}^2 \wedge \|\vec{x}\|_s = 1\} = \sup\{G(y) \mid y > 1\}$

is contained in the set $\{G(y_E), \lim_{y \rightarrow 1} G(y), \lim_{y \rightarrow \infty} G(y)\} = \{[1 + |b|^{\frac{s-t}{s-t}}]^{\frac{s-t}{s-t}}, |b|, 1\}$.

Again we consider three subcases.

Subcase 1: $s < t \Rightarrow M_{s,t} = 1$ and $\|\mathbf{D}\|(s, t) = |v_2| \cdot M_{s,t} = |v_2|$.

Subcase 2: $s > t \Rightarrow M_{s,t} = [1 + |b|^{\frac{s-t}{s-t}}]^{\frac{s-t}{s-t}}$ and $\|\mathbf{D}\|(s, t) = |v_2| \cdot M_{s,t} = [|v_2|^{\frac{s-t}{s-t}} + |v_1|^{\frac{s-t}{s-t}}]^{\frac{s-t}{s-t}}$.

Subcase 3: $s = t \Rightarrow$ We get $M_{s=t} = \lim_{y \rightarrow \infty} G(y) = 1$, hence $\|\mathbf{D}\|(s, t) = |v_2|$, and the theorem has been proved for $n = 2$. Now we finish **Case b** in a similar way to **Case a**.

Subcase 1: $s < t$ or $s = t$.

We have proved the theorem for $n = 2$. Because of $t < 0$, we have for arbitrary

$y_1, y_2, w_1, w_2 \in \mathbb{C}$: $|w_1 y_1|^t + |w_2 y_2|^t \geq [\max\{|w_1|, |w_2|\}]^t \cdot \left[\sqrt[s]{|y_1|^s + |y_2|^s} \right]^t$. Let $\mathbf{m} \in \{1, \dots, n-1\}$ with $|v_{\mathbf{m}}| := \max\{|v_1|, \dots, |v_{n-1}|\}$, let $\vec{x} := (x_1, \dots, x_{n-1}, x_n) \in \mathbb{C}^n$.

We assume the theorem for $n-1$, hence we have: $\sqrt[t]{\sum_{i=1}^{n-1} |v_i x_i|^t} \leq |v_{\mathbf{m}}| \cdot \sqrt[s]{\sum_{i=1}^{n-1} |x_i|^s}$.

Because of $t < 0$, this is equivalent to $\sum_{i=1}^{n-1} |v_i x_i|^t \geq |v_{\mathbf{m}}|^t \cdot \left[\sqrt[s]{\sum_{i=1}^{n-1} |x_i|^s} \right]^t$.

$$\begin{aligned} \text{Thus it follows } \|\mathbf{D}(\vec{x})\|_t^t &= \sum_{i=1}^{n-1} |v_i x_i|^t + |v_n x_n|^t \\ &\geq |v_{\mathbf{m}}|^t \cdot \left[\sqrt[s]{\sum_{i=1}^{n-1} |x_i|^s} \right]^t + |v_n x_n|^t \\ &\geq [\max\{|v_{\mathbf{m}}|, |v_n|\}]^t \cdot \left[\sqrt[s]{\sum_{i=1}^{n-1} |x_i|^s + |x_n|^s} \right]^t = |v_{\mathbf{M}}|^t \cdot \|\vec{x}\|_s^t. \end{aligned}$$

Because of $t < 0$, this is equivalent to $\|\mathbf{D}(\vec{x})\|_t \leq |v_{\mathbf{M}}| \cdot \|\vec{x}\|_s$. Hence $\|\mathbf{D}\|_{s,t} \leq |v_{\mathbf{M}}|$.

To check equality, take for all sufficient large $k \in \mathbb{N}$ (i.e. such that $2 - (1 - \frac{1}{k})^s > 0$)

the vector $\vec{a}_k := (a_{k,1}, \dots, a_{k,n})$ with $a_{k,\mathbf{M}} := \sqrt[s]{2 - (1 - \frac{1}{k})^s}$, and for every

$i \in \{1, 2, \dots, n\} \setminus \{\mathbf{M}\}$ take $a_{k,i} := q_k := \sqrt[s]{\frac{(1 - \frac{1}{k})^s - 1}{n-1}}$. We have for all such k : $\|\vec{a}_k\|_s = 1$, and because of $s, t < 0$, we get $\lim_{k \rightarrow \infty} (q_k) = \sqrt[s]{0} = +\infty$, hence $\lim_{k \rightarrow \infty} ((q_k)^t) = 0$,

$$\begin{aligned} \text{and } \|\mathbf{D}(\vec{a}_k)\|_t &= \sqrt[t]{|v_{\mathbf{M}}|^t \cdot \left[\sqrt[s]{2 - (1 - \frac{1}{k})^s} \right]^t + \sum_{i=1, \dots, n \wedge i \neq \mathbf{M}} |v_i|^t \cdot (q_k)^t} \\ &= |v_{\mathbf{M}}| \cdot \sqrt[t]{\left[\sqrt[s]{2 - (1 - \frac{1}{k})^s} \right]^t + (q_k)^t \cdot \sum_{i=1, \dots, n \wedge i \neq \mathbf{M}} \left| \frac{v_i}{v_{\mathbf{M}}} \right|^t}, \end{aligned}$$

hence $\lim_{k \rightarrow \infty} \|\mathbf{D}(\vec{a}_k)\|_t = |v_{\mathbf{M}}|$. Thus $\|\mathbf{D}\|_{s,t} = |v_{\mathbf{M}}|$.

Subcase 2: $s > t$.

Let $\vec{x} := (x_1, \dots, x_{n-1}, x_n) \in \mathbb{C}^n$. We have proved the theorem for $n = 2$, that means

$|y_1 w_1|^t + |y_2 w_2|^t \geq [|y_1|^{\frac{s-t}{s-t}} + |y_2|^{\frac{s-t}{s-t}}]^{\frac{s-t}{s}} \cdot \left[\sqrt[s]{|w_1|^s + |w_2|^s} \right]^t$ for $y_1, y_2, w_1, w_2 \in \mathbb{C}$.

Assume the theorem for $n-1$, hence (because of $t < 0$)

$\sum_{i=1}^{n-1} |v_i x_i|^t \geq \left[\sum_{i=1}^{n-1} |v_i|^{\frac{s-t}{s-t}} \right]^{\frac{s-t}{s}} \cdot \left[\sqrt[s]{\sum_{i=1}^{n-1} |x_i|^s} \right]^t$. By doing similar estimations as three times before, we get $\|\mathbf{D}(\vec{x})\|_t^t = \sum_{i=1}^{n-1} |v_i x_i|^t + |v_n x_n|^t \geq \left[\sum_{i=1}^{n-1} |v_i|^{\frac{s-t}{s-t}} \right]^{\frac{s-t}{s}} \cdot \left[\sqrt[s]{\sum_{i=1}^{n-1} |x_i|^s} \right]^t +$

$|v_n x_n|^t \geq \left[\sum_{i=1}^{n-1} |v_i|^{\frac{s-t}{s-t}} + |v_n|^{\frac{s-t}{s-t}} \right]^{\frac{s-t}{s}} \cdot \left[\sqrt[s]{\sum_{i=1}^{n-1} |x_i|^s + |x_n|^s} \right]^t = \left[\|\vec{v}\|_{\frac{s-t}{s-t}} \right]^t \cdot \|\vec{x}\|_s^t$.

Because of $t < 0$, this is equivalent to $\|\mathbf{D}(\vec{x})\|_t \leq \|\vec{v}\|_{\frac{s-t}{s-t}} \cdot \|\vec{x}\|_s$.

Hence $\|\mathbf{D}\|_{s,t} \leq \|\vec{v}\|_{\frac{s-t}{s-t}}$. To check equality, one can use the same vector as above, i.e.,

define for $i = 1, 2, \dots, n$: $r_i := \sqrt[s-t]{|v_i|^t}$, and $\vec{z} := \frac{1}{\sqrt[s]{\sum_{i=1}^n |v_i|^{\frac{s-t}{s-t}}}} \cdot (r_1, \dots, r_n)$.

Case c) Let $-\infty < t < 0 < s < \infty$.

The proof is similar as the proofs before and we will not explain it in all details.

In the case of $\prod_{i=1}^n v_i = 0$, in CASE (\mathbb{B}) we already have proved that $\|\mathbf{D}\|_{s,t} = 0$. Note that $\frac{s \cdot t}{s-t} < 0$, hence $[\prod_{i=1}^n v_i = 0 \Rightarrow \|\vec{v}\|_{\frac{s \cdot t}{s-t}} = 0]$ follows. Now assume $\prod_{i=1}^n v_i \neq 0$.

Proof for $n = 2$. As in **Case a**, we consider the 2×2 matrix $\mathbf{D} := \begin{pmatrix} v_1 & 0 \\ 0 & v_2 \end{pmatrix}$.

With $v_M = v_2$ and $b := v_1/v_2$ we have $\mathbf{D} = v_2 \cdot \begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix} =: v_2 \cdot \tilde{\mathbf{D}}$. One

has $\|\mathbf{D}\|(s, t) = |v_2| \cdot \|\tilde{\mathbf{D}}\|(s, t) = |v_2| \cdot \sup\{\|\tilde{\mathbf{D}}(\vec{x})\|_t \mid \vec{x} \in \mathbb{C}^2 \wedge \|\vec{x}\|_s = 1\}$. Again we consider the map $G^t(y) := (\|\tilde{\mathbf{D}}(\vec{x})\|_t)^t = y^t \cdot |b|^t + [\sqrt[1-y^s]{1-y^s}]^t$, (here for all y in the open interval $(0, 1)$). As in **Case a**, we have: $(G^t)'(y_E) = 0 \Leftrightarrow y_E = \sqrt[1+|b|^{\frac{s \cdot t}{t-s}}]{\frac{1}{1+|b|^{\frac{s \cdot t}{t-s}}}}$, which yields

a minimum for the map G^t , but a maximum for the map G , and we get the maximum $\max\{\|\tilde{\mathbf{D}}(\vec{x})\|_t \mid \vec{x} \in \mathbb{C}^2 \text{ and } \|\vec{x}\|_s = 1\} = \max\{G(y) \mid y \in [0, 1]\} = G(y_E)$.

As above, we have $G(y_E) = [1 + |b|^{\frac{s \cdot t}{s-t}}]^{\frac{s-t}{s}}$, and $\|\mathbf{D}\|_{s,t} = |v_2| \cdot G(y_E) = [|v_2|^{\frac{s \cdot t}{s-t}} + |v_1|^{\frac{s \cdot t}{s-t}}]^{\frac{s-t}{s}}$, and the theorem is proved for $n = 2$.

Because of $t < 0$, we have to continue as in **Case b**, subcase 2.

Let $\vec{x} := (x_1, \dots, x_{n-1}, x_n) \in \mathbb{C}^n$, and let $y_1, y_2, w_1, w_2 \in \mathbb{C}$.

We just have proved that $|y_1 w_1|^t + |y_2 w_2|^t \geq [|y_1|^{\frac{s \cdot t}{s-t}} + |y_2|^{\frac{s \cdot t}{s-t}}]^{\frac{s-t}{s}} \cdot [\sqrt[1+|w_1|^s + |w_2|^s]{1+|w_1|^s + |w_2|^s}]^t$ holds.

Assuming the theorem for $n-1$, we get $\sum_{i=1}^{n-1} |v_i x_i|^t \geq [\sum_{i=1}^{n-1} |v_i|^{\frac{s \cdot t}{s-t}}]^{\frac{s-t}{s}} \cdot [\sqrt[1+\sum_{i=1}^{n-1} |x_i|^s]{1+\sum_{i=1}^{n-1} |x_i|^s}]^t$.

Hence we compute as four times before

$$\begin{aligned} [\|\mathbf{D}(\vec{x})\|_t]^t &= \sum_{i=1}^{n-1} |v_i x_i|^t + |v_n x_n|^t \geq [\sum_{i=1}^{n-1} |v_i|^{\frac{s \cdot t}{s-t}}]^{\frac{s-t}{s}} \cdot [\sqrt[1+\sum_{i=1}^{n-1} |x_i|^s]{1+\sum_{i=1}^{n-1} |x_i|^s}]^t + |v_n x_n|^t \\ &\geq [\sum_{i=1}^{n-1} |v_i|^{\frac{s \cdot t}{s-t}} + |v_n|^{\frac{s \cdot t}{s-t}}]^{\frac{s-t}{s}} \cdot [\sqrt[1+\sum_{i=1}^{n-1} |x_i|^s + |x_n|^s]{1+\sum_{i=1}^{n-1} |x_i|^s + |x_n|^s}]^t = [\|\vec{v}\|_{\frac{s \cdot t}{s-t}}]^t \cdot \|\vec{x}\|_s^t. \end{aligned}$$

Because of $t < 0$, this is equivalent to $\|\mathbf{D}(\vec{x})\|_t \leq \|\vec{v}\|_{\frac{s \cdot t}{s-t}} \cdot \|\vec{x}\|_s$, hence $\|\mathbf{D}\|(s, t) \leq \|\vec{v}\|_{\frac{s \cdot t}{s-t}}$. To check equality, one can use the same vector as two times before, i.e. define for $i = 1, 2, \dots, n$: $r_i := \sqrt[1+|v_i|^s]{|v_i|^{\frac{s \cdot t}{s-t}}}$, and $\vec{z} := \frac{1}{\sqrt[1+\sum_{i=1}^n |v_i|^s]{1+\sum_{i=1}^n |v_i|^s}} \cdot (r_1, \dots, r_n)$.

Case d) Let $-\infty = s \leq t < 0$ and $\prod_{i=1}^n v_i \neq 0$, or let $0 < s \leq t = \infty$.

If $0 < s \leq t = \infty$, take $\vec{e}_M := (0, \dots, 0, 1, 0, \dots, 0)$, hence $\|\vec{e}_M\|_s = 1$, and $\|\mathbf{D}(\vec{e}_M)\|_\infty = |v_M|$, and $\|\mathbf{D}\| = |v_M| = \max\{|v_1|, \dots, |v_n|\}$ follows.

If $-\infty = s \leq t < 0$ and $\prod_{i=1}^n v_i \neq 0$ one can use the vector \vec{e}_k (for all $k \in \mathbb{N}$) with $e_{k,M} := 1$, and for all $i \in \{1, \dots, n\} \setminus \{M\}$ $e_{k,i} := k$, hence $\|\vec{e}_k\|_\infty = 1$, and $\lim_{k \rightarrow \infty} (\|\mathbf{D}(\vec{e}_k)\|_t) = |v_M| = \|\vec{v}\|_\infty$, and all four cases **Case a** – **Case d** are proved, hence CASE (\mathbb{C}) and CASE (\mathbb{D}) are confirmed.

It remains to prove one case of the theorem.

CASE (\mathbb{E}). Let $t = -\infty$ and $\prod_{i=1}^n v_i \neq 0$. As it has been shown before, the statement is true if $(t = -\infty = s)$ or $(t = -\infty \text{ and } s = \infty)$. So assume $t = -\infty < s \in \mathbb{R} \setminus \{0\}$. Take a $t \neq 0$ with $-\infty < t < s$, it is already proved that $\|\mathbf{D}\|_{s,t} = \|\vec{v}\|_{\frac{s \cdot t}{s-t}}$. Thus

$$\|\mathbf{D}\|_{s,-\infty} = \lim_{t \rightarrow -\infty} [\|\mathbf{D}\|_{s,t}] = \lim_{t \rightarrow -\infty} [\|\vec{v}\|_{\frac{s \cdot t}{s-t}}] = \|\vec{v}\|_{-s}.$$

For equality one takes the vector $\vec{z} := \|\vec{v}\|_{-s} \cdot (\frac{1}{v_1}, \dots, \frac{1}{v_n}) = \sqrt[1+\sum_{i=1}^n \frac{1}{|v_i|^s}]{1+\sum_{i=1}^n \frac{1}{|v_i|^s}} \cdot (\frac{1}{v_1}, \dots, \frac{1}{v_n})$, hence $\|\vec{z}\|_s = 1$ and $\|\mathbf{D}(\vec{z})\|_\infty = \|\vec{v}\|_{-s}$, and the proof of **Theorem 1** is finished.

3 Proofs of Theorem 2 and the Corollaries

The **Corollary 1** follows immediately by observing that

$$\frac{s \cdot t}{s-t} = \frac{(-t) \cdot (-s)}{(-t) - (-s)}, \quad \text{and} \quad s \leq t \iff -t \leq -s.$$

Before we can prove **Theorem 2** we mention a fact, which is easy to confirm.

Fact 1. *Let $r, s, t \in \mathbb{R}$, such that $0 \neq r \cdot s \cdot t$ and $\frac{1}{t} = \frac{1}{r} + \frac{1}{s}$. Then either $t < r, s$ or $t > r, s$. If furthermore $t < 0, r, s$ or $t > 0, r, s$, then $r \cdot s < 0$.*

Now we are able to prove **Theorem 2**.

Proof. This theorem is trivial if $n = 1$. So let $n > 1$. Let $t < r, s$. Now take the **Theorem 1**, CASE (C), and note that $r = \frac{s \cdot t}{s-t}$.

Let $t > r, s$. In the case of $\|\vec{v}\|_r \cdot \|\vec{x}\|_s = 0$, the inequality holds. Hence assume $\|\vec{v}\|_r \cdot \|\vec{x}\|_s \neq 0$. Because of **Fact 1** and $\frac{1}{t} = \frac{1}{r} + \frac{1}{s}$, three cases are possible, namely $0 > t > r, s$, or $t > r > 0 > s$, or $t > s > 0 > r$.

In the first two cases s is negative, and because of $\|\vec{x}\|_s \neq 0$, $x_i \neq 0$ holds for every i . One has $\frac{1}{r} = \frac{1}{t} + \frac{1}{-s}$ and (with **Fact 1**) $r < t, -s$. Let for all $i \in \{1, \dots, n\}$: $\tilde{x}_i := v_i \cdot x_i$ and $z_i := \frac{1}{x_i}$. Because of $r < t, -s$ we get

$$\sqrt[r]{\sum_{i=1}^n |\tilde{x}_i \cdot z_i|^r} \leq \sqrt[t]{\sum_{i=1}^n |\tilde{x}_i|^t} \cdot \sqrt[-s]{\sum_{i=1}^n |z_i|^{-s}} \iff \sqrt[r]{\sum_{i=1}^n |\tilde{x}_i \cdot z_i|^r} \cdot \sqrt[s]{\sum_{i=1}^n |z_i|^s} \leq \sqrt[t]{\sum_{i=1}^n |\tilde{x}_i|^t} \iff \|\vec{v}\|_r \cdot \|\vec{x}\|_s \leq \|\vec{v} \cdot \vec{x}\|_t.$$

The remaining last case $t > s > 0 > r$ is treated in the same way: because of $0 > r$ and $\|\vec{v}\|_r \neq 0$, $v_i \neq 0$ holds for every i . Hence define for all $i \in \{1, \dots, n\}$: $\tilde{x}_i := v_i \cdot x_i$ and $z_i := \frac{1}{v_i}$, and then one can go the same way as only just. This finishes the proof. \square

The **Corollary 2** follows directly from **Theorem 2**.

Remark 4. However, this version of the Hölder-inequality is not really an extension, but equivalent with the usual one ($\frac{1}{t} = \frac{1}{r} + \frac{1}{s}$ and $1 < r, s \implies \|\vec{v} \cdot \vec{x}\|_1 \leq \|\vec{v}\|_r \cdot \|\vec{x}\|_s$).

For positive values of r, s, t one can find a short proof in [1], p.103. The general case which includes negative values is treated in the next section.

4 Measurable Functions

In this last section we demonstrate that the generalized Hölder inequality also holds in the \mathcal{L}^p function spaces. The proofs rely mainly on the standard Hölder inequality. At first we have to define the \mathcal{L}^p spaces also for negative p .

Let $(\Omega, \mathcal{A}, \mu)$ be a measure space with $\mu(\Omega) > 0$. We use the conventions $\infty \cdot 0 := 0$ and $\frac{1}{0} := \infty$. Let $\mathcal{M}_\Omega := \{f : (\Omega, \mathcal{A}, \mu) \rightarrow \mathbb{R} \cup \{-\infty, \infty\} \mid f \text{ is measurable}\}$. Define for every $p < 0$: $\mathcal{L}^p := \mathcal{L}^\infty := \{f \in \mathcal{M}_\Omega \mid \text{ess sup } \{|f(\omega)| \mid \omega \in \Omega\} < \infty\}$.

Then we define for all $f \in \mathcal{M}_\Omega$

$$\|f\|_p := \begin{cases} \sqrt[p]{\int_\Omega |f|^p d\mu} & \iff 0 < \int_\Omega |f|^p d\mu < \infty \\ 0 & \iff \int_\Omega |f|^p d\mu = \infty \\ \infty & \iff \int_\Omega |f|^p d\mu = 0 \end{cases}$$

Note that for $f \in \mathcal{L}^\infty$, $\|f\|_p < \infty$ holds. And for every $p > 0$ we take the usual definition, $\mathcal{L}^p := \{f : (\Omega, \mathcal{A}, \mu) \rightarrow \mathbb{R} \cup \{-\infty, \infty\} \mid f \in \mathcal{M}_\Omega \text{ and } \int_\Omega |f|^p d\mu < \infty\}$, and for all $f \in \mathcal{M}_\Omega$ take $\|f\|_p := \sqrt[p]{\int_\Omega |f|^p d\mu}$.

By making an equivalence relation N ($f \approx_N g \Leftrightarrow f, g$ distinguish only on a zero set), and by defining $\mathbf{M}_\Omega := \mathcal{M}_\Omega / \approx_N$, and for all $p \in \mathbb{R} \setminus \{0\}$: $\mathbf{L}^p := \mathcal{L}^p / \approx_N$, this definition makes that the pairs $(\mathcal{M}_\Omega, \|\cdot\|_p)$, $(\mathbf{M}_\Omega, \|\cdot\|_p)$, $(\mathcal{L}^p, \|\cdot\|_p)$ and $(\mathbf{L}^p, \|\cdot\|_p)$ are **hw spaces** for all $p \in \mathbb{R} \setminus \{0\}$. These homogeneous weights $\|\cdot\|_p$ we call the Hölder weights on \mathcal{M}_Ω , \mathcal{L}^p , \mathbf{M}_Ω or \mathbf{L}^p , respectively. It is known that $(\mathbf{L}^p, \|\cdot\|_p)$ is a pseudonormed space if and only if $p > 0$, and it is a normed space if and only if $p \geq 1$.

Now let us recall the well-known Hölder inequality and the reverse Hölder inequality for measurable functions. For two real numbers r, s such that $1 < r, s$ and $1 = \frac{1}{r} + \frac{1}{s}$, we have for all measurable functions f, g (that means $f, g \in \mathcal{M}_\Omega$): $\|f \cdot g\|_1 \leq \|f\|_r \cdot \|g\|_s$.

For the next inequality see e.g. [2], p.226, or [3], p.51, or [4], p.191.

Corollary 3. Let $r, s \in \mathbb{R} \setminus \{0\}$ such that $1 > r, s$ and $1 = \frac{1}{r} + \frac{1}{s}$. (Hence either $r < 0 < s$ or $s < 0 < r$).

Then one has for all measurable functions f, g , that a reverse Hölder inequality holds, i.e.

$$\|f \cdot g\|_1 \geq \|f\|_r \cdot \|g\|_s.$$

Proof. Assume $r < 0 < s < 1$. Now we have to distinguish three cases.

1) $\|f\|_r = 0$. The inequality holds. (Note that $\infty \cdot 0 = 0$).

2) $\|f\|_r = \infty$. We have

$$\|f\|_r = \infty \iff \int_\Omega |f|^r d\mu = 0 \iff |f|(\omega) = \infty \text{ (for almost all } \omega \in \Omega).$$

In the case of $\|g\|_s = 0$, the inequality holds. In the case of $\|g\|_s > 0$, there is a measurable set A with $A \subset \Omega$, and $\mu(A) > 0$ and $|g|(\omega) > 0$ ($\forall \omega \in A$), hence it follows $|f \cdot g|(\omega) = \infty$ (for almost all $\omega \in A$), hence $\|f \cdot g\|_1 = \infty$.

3) $0 < \|f\|_r < \infty$.

We have $\frac{1}{s} = 1 + \frac{1}{-r}$, hence $1 = \frac{1}{1/s} + \frac{1}{-r/s}$ and (with **Fact 1**) $1 < \frac{1}{s}, \frac{-r}{s}$. Define $v := |f|^s \cdot |g|^s$, $w := |f|^{-s}$, hence $v, w \in \mathcal{M}_\Omega$, and we have by the Hölder inequality (note that $0 < \int_\Omega |w|^{\frac{-r}{s}} d\mu < \infty$)

$$\begin{aligned} \|v \cdot w\|_1 &\leq \|v\|_{\frac{1}{s}} \cdot \|w\|_{\frac{-r}{s}} \iff \sqrt[s]{\int_\Omega |v \cdot w| d\mu} \leq \int_\Omega |v|^{\frac{1}{s}} d\mu \cdot \sqrt[-r]{\int_\Omega |w|^{\frac{-r}{s}} d\mu} \\ &\iff \sqrt[s]{\int_\Omega |v \cdot w| d\mu} \cdot \sqrt[+r]{\int_\Omega |w|^{\frac{-r}{s}} d\mu} \leq \int_\Omega |v|^{\frac{1}{s}} d\mu \iff \|g\|_s \cdot \|f\|_r \leq \|f \cdot g\|_1 \end{aligned}$$

and all three cases of **Corollary 3** has been proved. \square

Now we are able to formulate the generalized Hölder inequality for measurable functions.

Theorem 3. Let $r, s, t \in \mathbb{R}$ such that $0 \neq r \cdot s \cdot t$ and $\frac{1}{t} = \frac{1}{r} + \frac{1}{s}$. Then we have for all $f, g \in \mathcal{M}_\Omega$

$$t < r, s \implies \|f \cdot g\|_t \leq \|f\|_r \cdot \|g\|_s.$$

$$t > r, s \implies \|f \cdot g\|_t \geq \|f\|_r \cdot \|g\|_s.$$

Proof. The proof is inspired by [1], p.103. We distinguish four cases.

1) $t < r, s$ and $t > 0$ 2) $t < r, s$ and $t < 0$

3) $t > r, s$ and $t > 0$ 4) $t > r, s$ and $t < 0$

We only show case 2. All the other cases follow along the same lines.

Let $t < r, s$ and $t < 0$.

Let $f, g \in \mathcal{M}_\Omega$. Then define $v, w \in \mathcal{M}_\Omega$, by taking $v := |f|^t$, $w := |g|^t$.

Because of $1 = \frac{1}{r/t} + \frac{1}{s/t}$, and $1 > \frac{r}{t}, \frac{s}{t}$, and because of the previous **Corollary 3**, we have $\|v \cdot w\|_1 \geq \|v\|_{\frac{r}{t}} \cdot \|w\|_{\frac{s}{t}} \iff \|f \cdot g\|_t \leq \|f\|_r \cdot \|g\|_s$. \square

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